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Polemic paper

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Explosive Behavior and Rational Bubbles: Evidence from the Serbian Hyperinflation at Daily Frequency

Summary: Using recently developed right-tailed sequential unit root tests at daily frequency in the extreme portion of the Serbian hyperinflation, we found that the money supply and the exchange rate exploded while the economy was approaching the maximum of the inflation tax Laffer curve, and remained explosive while on the wrong, decreasing side of that curve throughout the end of the hyperinflation. Money supply exploded as government tried first to raise seigniorage to a new higher plateau, and subsequently to prevent, albeit unsuccessfully, a decline in seigniorage while the economy was sliding down the decreasing side of the Laffer curve. The rational bubble was not found as the exchange rate explosiveness was driven by the explosiveness of its fundamental value: current and expected future money supply. Thus, even the extreme portion of the Serbian hyperinflation was driven by ever expanding money supply in the government quest for additional seigniorage and not by the rational bubbles.

Keywords: Explosiveness, Rational bubbles, Hyperinflation, Sequential unit root tests.

JEL: C12, C22, E31, F31.

The extreme Serbian hyperinflation – its causes, dynamics and political economy, as well as its spectacular halt, are thoroughly analysed in the special issue of *Panoeconomicus*, Volume 69, Issue 2. In a set of papers, it was vividly shown that by the end of the hyperinflation, exchange rate depreciation and money growth accelerated enormously reaching unprecedented levels (cf. Milojko Arsić 2022; Željko Bogetić, Diana Dragutinović, and Pavle Petrović 2022; Danica Popović 2022). The latter coincided with sharp decline in tax revenues (Dejan Popović 2022).

This paper analyse more closely aforementioned stylized facts by exploring whether the government had expanded money supply too far in search for ever increasing seigniorage, or rational speculative bubbles were present in the extreme portion of the Serbian hyperinflation, becoming its main driver.

It is in hyperinflation, with an extreme pace of money growth, inflation and currency depreciation, that one would expect, if anywhere, bubbles to appear. However, empirical findings on the presence of rational bubbles in hyperinflation are inconclusive (see Efthymios Pavlidis, Ivan Paya, and David Peel 2012). Moreover, empirical literature on the existence of asset price bubbles in general is indecisive as well (see Refet S. Gürkaynak 2008).

While examining the presence of bubbles, we shall use cointegration based tests (see Gürkaynak 2008), adapted for testing in hyperinflation (see Tom Engsted 2003). Nevertheless, in order to address the issue raised by George W. Evans (1991), that these tests can rule out only monotonically increasing bubbles but not the periodically collapsing ones, we have used recently developed sequential unit root tests, developed in the series of papers by Phillips with his coauthors (see Peter C. B. Phillips, Yangru Wu, and Jun Yu 2011; Phillips, Shuping Shi, and Yu 2014, 2015a, b), that can detect a temporary explosive process, as well as determine the corresponding explosive sub-periods. In addition, our large daily data set should alleviate the Evans's (1991) concern that, in a small sample (e.g. monthly ones for hyperinflation), just a few growing bubbles observations may appear and hence tests might not detect them. Sequential unit root tests have recently been used to test bubbles in the German hyperinflation (see Pavlidis, Paya, and Peel 2012), albeit by using 86 weekly observations and a different testing set-up.

The abundant daily data set (almost 150 observations) used in this paper enabled us to focus on the extreme portion of Serbian hyperinflation, quite a severe one, and in this extreme portion one would, if anywhere, expect bubbles to appear. Moreover, the abundant data set lends an opportunity to study the behavior around the maximum of the inflation tax Laffer curve. Both were infeasible in the prior hyperinflation studies at monthly frequency, due to small samples available.

1. Extreme Portion of the Serbian Hyperinflation: Background and a Structural Model for Testing Bubbles

The Serbian hyperinflation, that started in February 1992 and was halted in January 1994, was an extreme episode. It is the third highest in terms of its peak monthly inflation rate (after the Hungarian one of 1945-1946 and Zimbabwean hyperinflation), but the most extreme when the peak monthly inflation rate and its duration are considered together (Bogetić, Dragutinović, and Petrović 2022). The Serbian hyperinflation is thus by far more severe than the well-known German hyperinflation of the 1920s (cf. Petrović, Bogetić, and Zorica Vujošević (Mladenović) 1999; Bogetić, Dragutinović, and Petrović 2022; Popović 2022).

We shall examine the most extreme portion, i.e., the last seven months, of the severe Serbian hyperinflation at daily frequency. This period is characterized by an average monthly currency depreciation rate of 10,700% which is 33 times higher than the average inflation rate (322%)[1] in the German hyperinflation. Hence in this severe environment one would, if anywhere, expect to find explosive behaviour and possibly rational bubbles.

Table 1 details the development of the extreme portion of the Serbian hyperinflation at daily frequency by depicting the average daily rates of exchange rate depreciation (Δe) and money growth (Δm), as well as inflation tax (IT) and seignoirage (Sg). Figures 1 and 2 give the corresponding levels, i.e. the (logs of) money supply (m) and exchange rate (e).

[1] Both are discrete rates i.e. $(X_t / X_{t-1}) - 1$, usually reported for hyperinflation episodes including the German one (see Phillip D. Cagan 1956), and they are greater than log difference ones: $\ln X_t - \ln X_{t-1}$, the more so for the higher rates.

Table 1 Development of the Extreme Portion of the Serbian Hyperinflation (Daily Frequency)

Average per day	Δe	Δm	*IT*	*Sg*
July 1993	9.2%	8.1%	3.9	3.4
August	7.7	7.6	3.2	3.1
September	7.2	6.7	2.8	2.6
October	11.6	10.1	3.3	2.9
November	19.7	19.4	3.9	3.5
December 1st - 12th and 20th - 27th	36.4	33.4	3.0	2.9
January 94 11th - 21st	40.6	38.7	1.4	1.5

Notes: The rates above are defined as log difference, e.g. $\Delta e = \ln(E / E_{-1})$. In December and January, some observations for money supply are missing, and we have interpolated them, while reporting *IT* and *Sg* for sub-periods for which data is available. Data for January runs through Friday, January 21, since stabilization that halted hyperinflation was introduced on Monday, January 24 (cf. Arsić 2022; Bogetić, Dragutinović, and Petrović 2022; Ljubomir Madžar 2022). *IT* and *Sg* are averages per day expressed in million German marks (see Mladenović and Petrović 2010).

Source: Authors' calculations.

Average daily rates of currency depreciation and money growth were high, around 8%, and relatively stable from July through October, (almost) doubling in November and again in December, and running even higher in January 1994 (see Table 1 and Figures 1 and 2). Inflation tax (*IT*) and seigniorage (*Sg*) were also pushed up to a new higher plateau in November, but then they started to decline in December and January, despite further sharp increase in the rates of money growth and currency depreciation (see Table 1). This suggests that inflation tax and seigniorage exhibited the Laffer curve property, where *Sg* and *IT* reached respective maximum values sometime at the end of November or in early December 1993, and subsequently started to decrease, while switching to the wrong side of the Laffer curve.

A structural model of hyperinflation against which the presence of bubbles was tested is a standard monetary present value model of exchange rate, stating that:

$$e_t = (1 - a) \Sigma_{i=0}^{k} a^i E_t m_{t+i} - (1 - a) \Sigma_{i=0}^{k} a^i E_t u_{t+i} + a^{k+1} E_t e_{t+i+k}, \quad (1)$$

i.e. (log of) exchange rate (e) is determined by (log of) its sole fundamental in hyperinflation - money supply (m).

In a few steps one gets (cf. Engsted 1993):

$$S \equiv (m_t - e_t) + \alpha \Delta m_t = -(1- a)^{-1} \Sigma_{i=1}^{k} a^i E_t \Delta^2 m_{t+i} + (1 - a) \Sigma_{i=0}^{k} a^i E_t u_{t+i} - a^{k+1} E_t e_{t+i+k}, \quad (2)$$

where α is the Cagan money demand semi-elasticity, $a = \alpha / (1+\alpha)$ the discount factor, and u is money demand velocity shock. If there is a rational bubble the last term in (1) and (2) is explosive. Thus the presence of bubbles was explored by testing whether S is (temporary) explosive.

Using daily data set for the Serbian hyperinflation (referred to above) from July to December 1993 and standard cointegration tests, it has been obtained that both the Cagan model holds and S is stationary thus implying the absence of bubbles (see Mladenović and Petrović 2010). Nevertheless, since standard cointegration tests might suggest stationarity of S even though it was temporary explosive (see Evans 1991) we shall apply sequential unit root tests to examine whether this series exhibited temporary explosiveness, hence pointing to the presence of bubbles.

Estimated Cagan money demand semi-elasticity is 5.37 and it is stable i.e. does not significantly change from mid-November through December. Moreover, this estimate suggests that the maximum of the Laffer curve was attained sometime at the end of November or in early December 1993 as implied by Table 1 (Mladenović and Petrović 2010).

The data to be used are black-market exchange rates and the currency in circulation (cash) as money supply, both with a daily frequency. The source for the daily cash series is the central bank of Serbia, while black-market exchange rates were taken from newspaper where they were regularly recorded. The daily data for exchange rates is available for the whole period, i.e. through January 21, 1994. However some observations on the cash money supply are missing in December and January, (i.e. for December 13-18, and December 28-January 10 periods), so we have interpolated them, and used this money series when analysing the whole period. All reported estimates are based on samples that cover five working days per week, since the money supply did not change over weekends.

2. Empirical Findings

As motivated in Section 1, we tested for temporary explosiveness of (logs of) money supply (m), exchange rate (e), and the exchange rate corrected for its fundamental values: preliminary with m and eventually with $m + \alpha\Delta m$. This is also in line with Timo Bettendorf and Wenjuan Chen (2013) who likewise tested the presence of the exchange rate bubble by subtracting alternative fundamentals from the exchange rate.

Recently advanced sequential ADF tests for collapsing bubbles: GSADF and SADF (see the Appendix) were used to examine whether these series were (temporary) explosive and, if so, for which sub-periods. Under the null hypothesis the series was I(1) while explosive under the alternative, hence implying a right-tailed test. The sample used runs from July 1, 1993 to January 21, 1994, containing 147 observations. The testing results that examined the presence of temporary explosiveness are presented in Table 2 below, while Figures 1 to 4 show for each variable explosive sub-periods, if any.

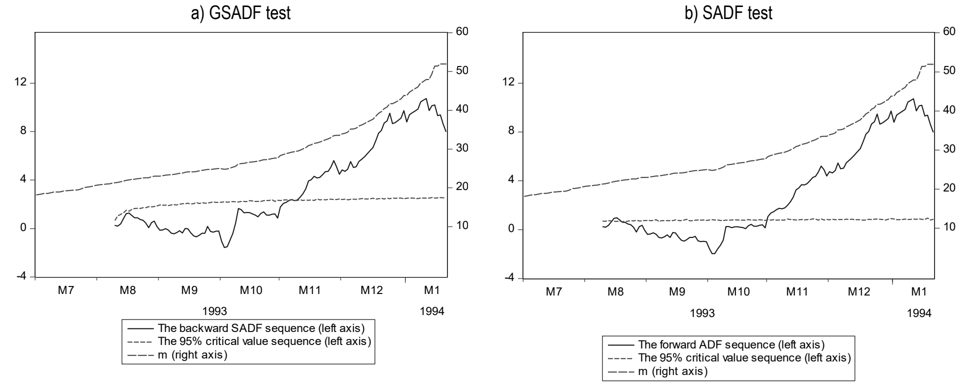
Table 2 Temporary Explosiveness: Testing Results; Sample: July 1, 1993 - January 21, 1994

Variable	GSADF	Critical values 1%, 5% and 10%	SADF	Critical values 1%, 5% and 10%
e_t	5.17	3.22, 2.60, 2.29	5.09	2.59, 2.04, 1.71
m_t	10.73	3.08, 2.56, 2.26	10.73	2.51, 1.97, 1.64
$(e_t - m_t)$	2.66	2.42, 1.92, 1.63	2.51	1.87, 1.22, 0.92
$S_t \equiv (m_t - e_t) + 5.37\,\Delta m_t$	-1.58	4.26, 3.68, 3.38	-3.78	3.23, 2.63, 2.32

Notes: The initial window size was set as 0.2*T, and T is the sample size, which is 29 for the SADF and GSADF tests. Critical values were obtained from Monte-Carlo simulations with 5,000 replications for the SADF and 2000 replications for GSADF tests. The number of lags is equal to 5 for e_t, 0 for m_t, 6 for $(e_t - m_t)$ and 0 for S_t. Simulations also took into account estimates of the drift term.

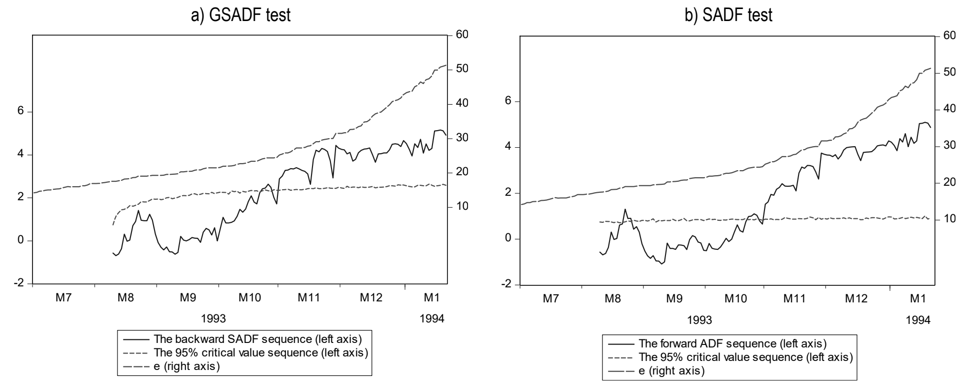
Source: Authors' calculations.

Testing results reported in Table 2 show that the exchange rate (e_t), money supply (m_t) and the exchange rate corrected with current money supply ($e_t - m_t$) were all temporary explosive (H_1) even at 1% significance level. However, S_t was non-explosive (H_0) still at 10% significance level.



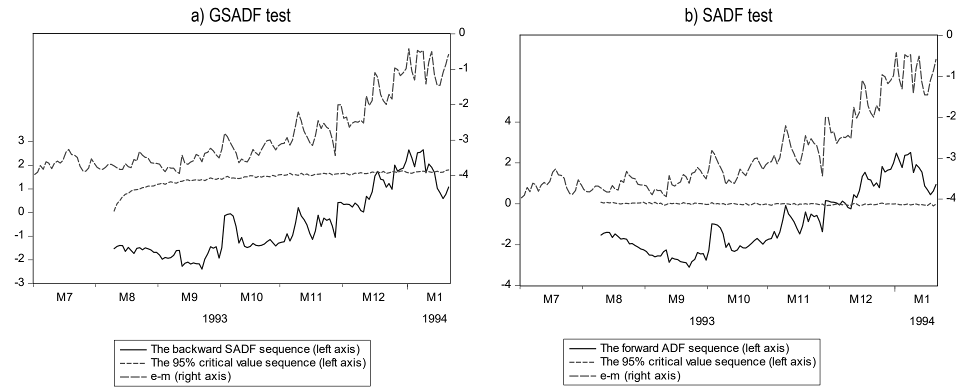
Source: Authors' calculations.

Figure 1 The Money Supply *m*



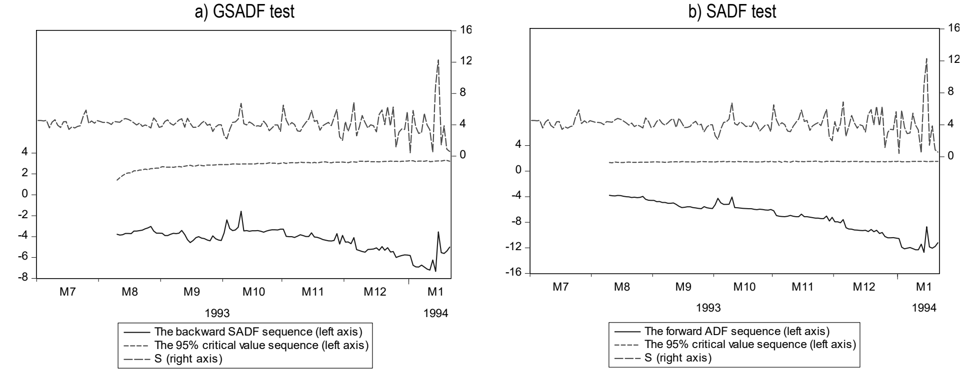
Source: Authors' calculations.

Figure 2 The Exchange Rate *e*



Source: Authors' calculations.

Figure 3 The Exchange Rate Corrected for Money Supply: (e – m)



Source: Authors' calculations.

Figure 4 The Exchange Rate Corrected for Its Fundamental Value: $S \equiv (m_t - e_t) + 5.37\Delta m_t$

The results reported above are summarised in Table 3 presenting the date when each series exploded and the period throughout which it was explosive.

Table 3 Temporary Explosiveness of the Series: The Starting Date and the Period

	GSADF test	SADF test
Money supply: m	November 10, 1993 onwards	November 1, 1993 onwards
Exchange rate: e	November 1, 1993 onwards	November 1, 1993 onwards
$(e - m)$	December 27, 1993 - January 14, 1994	December 13, 1993 onwards
$-S \equiv (e - m - 5.37\Delta m)$	Non-explosive	Non-explosive

Source: Authors' compilation.

The obtained results are robust, as both tests offer the same findings. Thus, both GSADF and SADF indicate that the three series: m, e and $(e - m)$ were temporary explosive, while S was not. The money supply and the exchange rate exploded practically at the same time, i.e. at the beginning of November and remained explosive through the end of the hyperinflation.

The exchange rate corrected preliminary by the money supply as its fundamental determinant: $(e - m)$, was still temporary explosive, for a much shorter period though: second half of December and most of January. Although the exact dates and periods differed a bit across the two tests, the qualitative findings are the same: $(e - m)$ exploded when hyperinflation was reaching its peak and the economy was sliding down the decreasing side of the Laffer curve.

Nevertheless, (temporary) explosiveness of $(e - m)$ does not still confirm the presence of rational bubbles. Eventually, one should test whether S was temporary explosive (cf. Equation (2) above), and we found that it was not, hence ruling out the presence of rational bubbles in the extreme portion of the Serbian hyperinflation.

3. Summary of Results and Conclusions

The exchange rate, although temporary explosive, did not exhibit rational bubbles in the Serbian hyperinflation, as its explosiveness was driven by market fundamentals.

Money supply (*m*) and the exchange rate (*e*) became temporary explosive towards the end of the Serbian hyperinflation when the latter was heading to its peak. More specifically, the two series: *m* and *e* exploded at the beginning of November 1993 when economy was approaching the maximum of the inflation tax Laffer curve and remained explosive through the end of hyperinflation (December and January) when the economy was on the decreasing side of the Laffer curve.

Explosion of the money supply could be explained by government's behaviour towards the end of hyperinflation specifically around the maximum of the Laffer curve. In this period, government sharply increased money supply attempting first to raise seigniorage to a new higher plateau, and subsequently to prevent, albeit unsuccessfully, seigniorage decline when economy switched to the decreasing side of the Laffer curve. Moreover, the exchange rate exploded at the same time as money supply did.

These results concur with Robert J. Barro's (1972, p. 979) suggestion "… that hyperinflations tend to become unstable (from the money-supply side) when the revenue-maximizing rate is exceeded". Nevertheless his proposal that money supply and inflation became explosive when their growth rates surpass inflation tax maximizing rate, is based on just a few monthly observations for the German (two months) and the Hungarian (four months) hyperinflations (cf. Barro 1972, p. 990). Yet by employing the abundant daily data set and newly developed sequential unit root tests that can detect temporary explosiveness we were able to confirm Barro's conjecture.

When the exchange rate is corrected for the money supply, i.e. its fundamental value, the adjusted exchange rate ($e - m$) became non-explosive in November and most of December (see Table 3). This indicates that the explosiveness of the exchange rate in this period was triggered and (dominantly) driven by explosiveness of current money supply.

Nevertheless, even the exchange rate corrected for its fundamental value: money supply, i.e. ($e - m$) temporary exploded at the end of December and in January, suggesting the presence of rational bubbles. Therefore, it eventually takes a structural model to test whether rational bubbles were present, specifically whether $-S_t \equiv e_t - (m_t + \alpha \Delta m_t)$ was (temporary) explosive (see Equation (2)). It turned out that it was not explosive (see Figures 4a and b and Table 2) thus ruling out the presence of even periodically collapsing rational speculative bubbles in the extreme portion of the Serbian hyperinflation. It follows that even the extreme portion of the Serbian hyperinflation was driven by ever expanding money supply in the government quest for additional seigniorage and not by the rational bubbles.

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Appendix

Sequential Right-Tailed Unit Root Tests: Supremum ADF (SADF) Test and the Rolling Window Generalized SADF Test (GSADF)

In Phillips, Wu, and Yu (2011) - PWY, and Phillips, Shi, and Yu (2014, 2015a and 2015b) - PSY testing procedure is developed to test the null hypothesis of a random walk with an asymptotically negligible drift against the alternative of time series following a random walk with explosive episodes.

In order to discriminate between two hypotheses recursive algorithm has been developed that extends the standard model for ADF test. Recursive estimation is based on a rolling window. Let us suppose that the rolling window regression sample begins at the r_1^{th} fraction of the sample T and ends at the r_2^{th} fraction of the sample such that $r_2 = r_1 + r_w$, where r_w denotes the window size of the regression. The baseline regression model reads as follows:

$$\Delta X_t = \alpha_{r_1,r_2} + \beta_{r_1,r_2} X_{t-1} + \sum_{i=1}^{k} \delta_{r_1,r_2}^{i} \Delta X_{t-i} + \varepsilon_t, \;\; \varepsilon_t : \mathrm{iid}(0, \sigma_{r_1,r_2}^2).$$

The order of the test is k.

When $k = 0$, this regression is estimated on sample $T_w = [T r_w]$. The ADF test calculated from this regression is denoted as $ADF_{r_1}^{r_2}$.

Two tests are proposed: supremum ADF (SADF) test and the rolling window generalized SADF test (GSADF). They are briefly overviewed.

The SADF test is based on estimating the ADF model using a sequence of forward expanding sample and identifying the sup value of the calculated ADF statistic sequence. The window size r_w enlarges from the smallest size r_0 to 1. Each sample begins at r_1 that is fixed at 0 and it ends at point r_2 that is equal to r_w. Starting point of the sample changes from r_0 to 1. ADF statistic obtained from sample that moves from 0 to r_w is denotes $ADF_0^{r_2}$. It is now possible to define SADF statistic as follows:

$$SADF(r_0) = \sup_{r_2 \in [r_0, 1]} ADF_0^{r_2}.$$

The GSADF test also consists of the recursive estimation of ADF test, but the type and size of subsamples differ. The GSADF test allows not only that the end point r_2 changes (from r_0 to 1), but also that the starting point r_1 moves within an interval that runs from 0 to r_2-r_0. The GSADF test is introduced as the maximum value of ADF statistic computed over all feasible values of r_1 to r_2. It is constructed for initial value r_0 as:

$$GSADF(r_0) = \sup_{\substack{r_2 \in [r_0, 1] \\ r_1 \in [0, r_2 - r_0]}} ADF_{r_1}^{r_2}.$$

The limiting distributions of SADF and GSADF tests are defined by PWY and PSY.

Upon the conclusion that time series has experienced explosive episodes, it is necessary to identify the beginning and the end of each explosive path. The question asked is whether given observation belongs to explosive trajectory. The so-called data-stamping procedure has been developed. It can be conducted in two ways.

The first strategy proposed by PWY consists of computing right-tailed ADF recursively. In this way the sequence ADF_{r2} is obtained with r_2 taking values from r_0 to 1. These values are compared with the right tailed critical value of ADF test. The beginning of explosive episode is detected as the first time series observation with ADF statistic greater than the critical value. This date is denoted as $[T\hat{r}_b]$. The end of the explosive episode is defined as $[T\hat{r}_e]$, being the first time series observation for which calculated ADF statistic over the window r_2 is lower than the critical value. The explosive episode is well defined if its duration is no less than $ln(T)$. The estimated (fractional) origin and the end of the explosive episode can be specified as follows:

$$\hat{r}_b = \inf_{r_2 \in [r_0,1]} \left\{r_2 \colon ADF_{r_2} > cv_{r_2}^{k\%}\right\},$$
$$\hat{r}_e = \inf_{r_2 \in [\hat{r}_b + ln(T)/T,1]} \left\{r_2 \colon ADF_{r_2} < cv_{r_2}^{k\%}\right\}$$

where $cv_{r_2}^{k\%}$ denotes the *100(1-k)%* critical value of the ADF test based on $[Tr_2]$ observations and the chosen significance level *k%.*

The second approach advanced by PSY employs the backward sup ADF test. The backward SADF test is defined as a sup ADF test that is computed on a sequence of backward expanding samples. This procedure assumes that each sample ends at fixed value r_2, whereas the sample starting point runs from 0 to r_2-r_0. The computed ADF statistic sequence is denoted as $\left\{ADF_{r_1}^{r_2}\right\}_{r_1 \in [0, r_2 - r_0]}$. The backward SADF statistic is defined as the maximum value of these values over the given interval:

$$BSADF_{r_2}(r_0) = \sup_{r_1 \in [0, r_2 - r_0]} \left\{ADF_{r_1}^{r_2}\right\}.$$

Explosive trajectory begins at the first observation for which backward sup ADF statistic is greater than the critical value of the backward sup ADF test. Explosive trajectory terminates at the first observation after $[T\hat{r}_b] + qln(T)$ for which BSADF statistic drops below the critical value of the backward sup ADF statistic. For the explosive episode to be well defined it should last longer than $qln(T)$, where q denotes frequency dependent parameter. The estimated beginning and termination of the explosive episode can be calculated as follows:

$$\hat{r}_b = \inf_{r_2 \in [r_0,1]} \left\{r_2 \colon BSADF_{r_2}(r_0) > scv_{r_2}^{k\%}\right\},$$
$$\hat{r}_e = \inf_{r_2 \in [\hat{r}_b + q\, ln(T)/T,1]} \left\{r_2 \colon BSADF_{r_2}(r_0) < scv_{r_2}^{k\%}\right\},$$

and $scv_{r_2}^{k\%}$ represents the *100(1-k)%* critical value of the sup ADF test based on $[Tr_2]$ observations and the chosen significance level *k%.*

We can make a closer connection with supremum ADF (SADF) test and the rolling window generalized SADF test (GSADF). Given that SADF test employs ADF test for each $r_2 \in [r_0, 1]$, it can be represented as follows:

$$SADF(r_0) = \sup_{r_2 \in [r_0,1]} \left\{ADF_{r_2}\right\}.$$

Similarly, the GSADF test relies on backward sup ADF for each r_2 running from r_0 to 1, so that the conclusion is derived from the sup values of the backward sup ADF statistic sequence. This can be written in the following way:

$$GSADF(r_0) = \sup_{r_2 \in [r_0,1]} \left\{BSADF_{r_2}(r_0)\right\}.$$

PSY provided the asymptotic properties of the data-stamping procedures. Under the presence of one explosive episode both procedures estimate the beginning and the termination dates consistently under general conditions. If two explosive paths exist, then only the BSADF algorithm identifies them consistently regardless of their relative duration.